Calculus on Manifolds

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A *n-dimensional topological manifold* M is a second countable, Hausdorff toplogical space such that for every $x \in M$ there is a neighborhood U of x such that U is homeomorphic to $\mathbb{R}^n$. A *k-smooth atlas* on M is a collection $\{(U_\alpha, \phi_\alpha)\}$ such that (a) $\{U_\alpha\}$ covers M and for each α, β, the function $\varphi_\beta \circ \varphi_\alpha^{-1}$ restricted to $\varphi(U_\alpha \cap U_\beta) \to \mathbb{R}^n$ is C^k. An *n-dimensional C^k manifold* is an n-dimensional topological manifold equipped with a C^k-smooth atlas. We say M is a *smooth manfiold* if $k = \infty$.

Let M and N be smooth manifolds of dimensions m and n, respectively, and let $f : M \to N$ be a function. Let $x_0 \in M$. Let (U, φ) be a chart containing x_0 and let (V, ψ) be a chart containing $f(x_0)$. Then the function $\psi \circ f \circ \varphi^{-1}$ is a function from $\varphi(U) \subset \mathbb{R}^m$ to $\psi(V) \subset \mathbb{R}^n$, and we say f is *C^k-smooth* at x_0 if the map $\psi \circ f \circ \varphi^{-1}$ is smooth at $\varphi(x_0)$. It is worth noting that this definition is independent of the charts chosen. Indeed, if (U', α) and (V', β) are two other charts of x_0 and $f(x_0)$, respectively, then by the fact that all charts are diffeomorphism, we get that $\beta \circ f \circ \alpha^{-1}$ is also smooth.

We begin by discussing paths on a manifold. Let $x_0 \in M$, and let $\gamma : (-1, 1) \to M$ be a smooth function such that $\gamma(0) = x_0$. Given such a gamma, we define $v_\gamma : C^\infty(M) \to \mathbb{R}$ by

$$v_\gamma(f) = (\frac{d}{dt} f \circ \gamma)(0).$$

Note that v_γ is linear, and if $f, g \in C^\infty(M)$ then

$$v_\gamma(fg) = f(p)v_\gamma(g) + v_\gamma(f)g(p)$$

i.e. v_γ satisfies the *Liebniz rule*. We recall that a *derivation* is a linear map satisfying the Liebniz rule. So each v_γ is a derivation. We say two curves γ_1 and γ_2 which pass throgh p are equivalent if for every $f \in C^\infty(M)$ we have

$$v_{\gamma_1}(f) = v_{\gamma_2}(f)$$

and we define the *tangent space at p* T_pM to be the equivalence classes of these curves. Elements of T_pM will be denoted by v_γ where γ is a curve through p.

Now fix a chart (U, x) of p where $x = (x^1, \ldots, x^n) : U \to \mathbb{R}^n$. Then a smooth function $f : M \to N$ can be written as $f(x^1, \ldots, x^n)$. Thus we can define the derivations $\frac{\partial}{\partial x^i}(p) : C^\infty(M) \to \mathbb{R}$ as follows:

$$\frac{\partial}{\partial x^i}(f)(p) := \frac{d}{dt}(f(x^1, \ldots, x^i + t, \ldots, x^n).$$

Thus, if γ is a curve going through p and we define the coordinates $x^i(t) = x^i(\gamma(t))$ then

$$(\frac{d}{dt} f \circ \gamma)(0) = \frac{d}{dt} f(x^1(t), \ldots, x^n(t)) \tag{1}$$

$$= \sum \frac{dx^i(t)}{dt} \frac{\partial f}{\partial x^i}(p) \tag{2}$$

We note also that the derivations $\partial^i(p) = \frac{\partial}{\partial x^i}(p)$ are linearly independent. Indeed, if $\sum a_i \partial^i = 0$ then applying this to x^j yields $a_j = 0$. Thus, we can identify T_pM with the real span of $\{\partial^1(p), \ldots, \partial^n(p)\}$, which has dimension n. Similar to the tangent space, we define the *co-tangent space* to be the dual of T_pM, i.e. T_p^*M. Given an n-dimensional smooth manifold M, we define its *tangent bundle* TM as

$$TM = \bigsqcup_{p \in M} T_pM$$

together with a projection map $\pi : TM \to M$ given by $\pi(v_p) = p$. We now define a manifold structure on TM. Let (U, φ) be a chart in M. Then for every $p \in U$, we can write $v_p = \sum v^i \frac{\partial}{\partial x^i}(p)$. Then we can define a chart $(\pi^{-1}(M), \tilde{\varphi})$ by

$$\tilde{\varphi}(v_p) = (\varphi(p), (v^1, \ldots, v^n)).$$

This makes TM into a $2n$-dimensional manifold.

We are now ready to define the differential of a function, which is the analog of the Jacobian for manifolds. Let $f : M \to N$ be a smooth map of manifolds, and fix a $p \in M$. Let v_a be a member of T_pM. Then if $g \in C^\infty(N)$ then we can define the *pushforward* or *differential* $f_{*,p} : T_pM \to T_{f(p)}N$ by

$$df := f_{*,p}(v_a)(g) := v_a(g \circ f).$$

As we have this for each p, we get a map between tangent bundles

$$f_* : TM \to TN$$

which restricts to $f_{*,p}$ on each T_pM. As it turns out, f is smooth if and only if f_* is smooth. We also denote f_* by df. We will define the d operator shortly. Similar to the tangent bundle, we define the *co-tangent bundle* which is defined as

$$T^*M = \bigsqcup_{p\in M} T_p^*M$$

which carries the "same" manifold structure as TM.

With the differential of a function defined, we turn our attention to k-forms, which are the appropriate domain of integration. Recall that for a vector space V, we can form its *tensor* and *exterior* algebras TV and ΛV, which have graded components T^kV and Λ^kV, respectively. Let π be the projection map from Λ^kT^*M onto M, induces from the projection $TM \to M$. For $k \geq 0$, a *differential k-form* ω is a smooth map $\omega : M \to \Lambda^kTM$ such that $\pi \circ \omega = \mathrm{id}_M$. First, note that it is clear that for each $p \in M$ we get an alternating form

$$\omega_p : (T_pM)^k \to \mathbb{R}.$$

Let (U, φ) be a local chart, let $\{\frac{\partial}{\partial x^i}\}$ be the basis of T_pM and let $\{dx^i\}$ be the basis of $(T_pM)^*$. Then we can write

$$\omega = \sum \omega_{i_1,\ldots,i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

where the sum ranges over all subsets of cardinality k of $\{x^1, \ldots, x^n\}$. We then let $\Omega^k(M)$ denote the k-forms on M, where we define $\Omega^0(M) = C^\infty(M)$. It is clear that $\Omega^r(M) = 0$ when $r > n$. If M is n-dimensional, then a nowhere vanishing n-form is called a *volume form*. Such a form ω has the form $\omega = f dx^1 \wedge \cdots dx^n$, where $f \in C^\infty(M)$ is such that $f(x) \neq 0$ for all $x \in M$. The intermediate value theorem implies that f has the same sign on all of M. We call this the *orientation* of the manifold.

Now let M be an n-dimensional manifold, and let $\omega \in \Omega^{n-1}(M)$. We want to find a formula for integrating ω as in Stoke's thereom from multivariable calculus. Let (U, φ) be a chart. Then we obtain a map $\omega^* = \omega \circ \phi^{-1}$, which is a $n-1$ form on $\varphi(U)$. Thus,

$$\int_{\varphi(U)} \omega \circ \varphi^{-1} = \int_{\partial\varphi(U)} d(\omega \circ \varphi^{-1}).$$

Let $(U_\alpha, \varphi_\alpha)$ be a countable atlas of M and let $\{\rho_\alpha\}$ be a partition of unity subordinate to $\{U_\alpha\}$. One can then write $\omega = \sum_\alpha \rho_\alpha \omega$. Then

$$\int_M \omega = \int_M \sum \rho_\alpha \omega = \sum \int_M \rho_\alpha \omega.$$

By Stoke's thereom in multivariable calculus, we obtain

$$\int_M \rho_\alpha \omega := \int_{\varphi(U)} \omega \circ \varphi^{-1} = \int_{\partial\varphi(U)} d(\omega \circ \varphi^{-1})$$

so

$$\int_M \omega = \sum \int_{\partial\varphi(U)} d(\omega \circ \varphi^{-1}) = \sum \int_{\partial U_\alpha} d\omega = \int_{\partial M} d\omega.$$

Theorem 1. *(Stoke's Theorem) Let M be a real orientable smooth manifold of dimension n with boundary, and let $\omega \in \Omega^{n-1}(M)$ be a differential form with compact support. Then*

$$\int_M d\omega = \int_{\partial M} \omega.$$

Let us end this section by defining the d operator that we defined on smooth functions. So far, we have one function $d : C^\infty(M) \to \Omega^1(M)$. We aim to define d as a map from $\Omega^k(M)$ to $\Omega^{k+1}(M)$. Let us assume that d satisfies the following:

1. d is linear
2. $d^2 = d \circ d = 0$
3. If $\omega \in \Omega^k(M)$ and $\eta \in \Omega^l(M)$ then $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$

Given our first d on smooth maps, it is clear that we can extend to a map from $\Omega^k(M)$ to $\Omega^{k+1}(M)$. This defines the *de Rham complex*, which is the cochain complex

$$\cdots \to \Omega^k(M) \xrightarrow{d_k} \Omega^{k+1}(M) \xrightarrow{d_{k+1}} \cdots$$

from which we obtain the *de Rham cohomology groups*

$$H^k_{DR}(M) = \ker(d_n)/\mathrm{im}(d_{n-1}).$$