Geometry and Applications

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We use the Riemann integral as the definition of the integral for most of this document, unless stated otherwise. At the end of this section we will briefly define the Lebesgue integral. We outline the Riemann integral because the propositions (change of variables, inverse function theorem, and Fubini's theorem) are harder to state in the measure-theory setting, and can be demonstrated by elementary arguments (where we omit the details).

Definition 1. *Let $A = [a_1, b_1] \times \cdots \times [a_n, b_n]$ be a rectangle in $\mathbb{R}^n$ and let $f : A \to \mathbb{R}$ be a bounded function. If for every $\varepsilon > 0$ there exist a partition P of rectangles $A_1, \ldots, A_k$ of A such that*

$$L = \sum_{i=1}^{k} \inf f(A_i)\, Vol(A_i) < \infty \quad and \quad U = \sum_{i=1}^{k} \sup f(A_i)\, Vol(A_i) < \infty \quad and \quad U - L < \varepsilon$$

then we say f is Riemann-integrable *on A. This is equivalent to*

$$\sup\{L(P, f)\} = \inf\{U(P, f)\}$$

which ranges over all partitions of A. In this case, the quantity above is called the integral of f over A *and we denote it by*

$$\int_A f.$$

Proposition 1. *(Change of variables)*

Proposition 2. *(Inverse function theorem)*

Proposition 3. *(Fubini's Theorem)*

Definition 2. *(Measure space, measurable set, measurable function)*

Definition 3. *(Lebesgue measure)*

Definition 4. *(Lebesgue integral, lebesgue integrable functions)*

Definition 5. *(L^p) For $0 < p < \infty$, we define*

$$L^p(X) = \left\{ \int_X |f|^p d\mu < \infty : f \in \mathcal{M}(X) \right\}.$$

4 Differential forms

Our first goal is reframe multivariable calculus in the language of differential forms. To this end, let $U \subset \mathbb{R}^n$ be an open subset. We define $C^\infty(U)$ to be the set of smooth functions from U to $\mathbb{R}$. Let $x_1, \ldots, x_n$ be a basis of $\mathbb{R}^n$, and let $x^1, \ldots, x^n$ be a dual basis. We note that the basis of the k-th tensor power $T^k\mathbb{R}^n$ has a basis of $x_{i_1} \otimes \cdots \otimes x_{i_k}$ where $\{i_1, \ldots, i_k\}$ are the subsets of $\{x_1, \ldots, x_n\}$ of cardinality k. We note (define?) $T^0\mathbb{R}^n = \mathbb{R}$. We then define the *tensor algebra*

$$T\mathbb{R}^n = \bigoplus_{k=0}^{\infty} T^k\mathbb{R}^n.$$

We note that this really is an algebra (we can concatenate tensors). Finally, we define the *exterior algebra* $\Lambda\mathbb{R}^n$ to be the quotient of $T\mathbb{R}^n$ by the two-sided algebra ideal generated by $\{v \otimes w - w \otimes v\}$. We let $\Lambda^k\mathbb{R}^n$ to be the k-th graded component of $\Lambda\mathbb{R}^n$.

We are now ready to define k-forms and the d operator. For $k = 0$, $\Omega^0(U) = C^\infty(U)$. For $k > 0$, we define $\Omega^k(U)$ to be the free $C^\infty(U)$ module generated by $\{dx^{i_1} \cdots dx^{i_k} : 1 \le i_1 < \cdots < i_k < n\}$. For $f \in C^\infty(U)$, we define

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i.$$

For $k > 1$, we define

$$d(\sum_\alpha \omega_\alpha dx^\alpha) = \sum_\alpha dw_\alpha \wedge dx^\alpha.$$

We first note that d is a linear map from $\Omega^k(U)$ to $\Omega^{k+1}(U)$. Secondly. we notice that when we expand dw_α using the chain rule, we see that the coefficients of the above equation have the form

$$\frac{\partial \omega_\alpha}{\partial x^j} \qquad \text{for some } j\ .$$

The Clairaut-Schwarz theorem, which states that mixed partions of smooth functions are equal, implies that $d^2 = 0$.

Let us step back and formulate a classical problem in vector calculus in the language of differential forms. Let $F = \mathbb{R}^3 \to \mathbb{R}$ be a smooth function which can be expressed as $F(x) = (P(x), Q(x), R(x))$ where P, Q and R are smooth functions on $\mathbb{R}^3$. Now let $\omega = Pdx + Qdy + Rdz$ be a 1-form on $\mathbb{R}^3$. Applying the formulas above yield

$$d\omega = (Q_x - P_y)dx \wedge dy + (R_x - P_z)dx \wedge dz + (R_y - Q_z)dy \wedge dz.$$

Notice that the coefficients of $d\omega$ are exactly the components of the *curl* of F.

Quickly, notice that differential forms are the sort of things we integrate. Thus if $f \in C^k(\mathbb{R}^n$, then we define

$$\int f dx^{i_1} \wedge \cdots \wedge dx^{i_k} = \int_{\mathbb{R}^n} f(\mathbf{x}) dx^{i_1} \cdots dx^{i_k}$$

where the right-hand side can be taken to be the Riemann-integral.

Now for k-forms $\alpha = \alpha_I dx^I$ and $\beta = \beta_I dx^I$, we define their inner product as

$$\langle \alpha, \beta \rangle = \sum \alpha_I \beta_I.$$

This restricts to the usual inner product on $\mathbb{R}^n$. Let us restrict our attention to $n = 3$ for a bit. We define the *canonical volume form* dV on $\mathbb{R}^3$ by $dV = dx \wedge dy \wedge dz$.

Proposition 4. *Let $U \subset \mathbb{R}^3$ and let $\alpha, \beta \in \Omega^k(U)$. Then there exists a unique 3-form $*\beta$ satisfying*

$$\alpha \wedge *\beta = \langle \alpha, \beta \rangle dV.$$

Proof. This can be verified as follows:

$$\begin{aligned} *1 &= dx \wedge dy \wedge dz \\ *dx &= dy \wedge dz \\ *dy &= dz \wedge dx \\ *dz &= dx \wedge dy \\ *(dy \wedge dz) &= dx \\ *(dz \wedge dx) &= dy \\ *(dx \wedge dy) &= dz \\ *(dx \wedge dy \wedge dz) &= 1 \end{aligned}$$

First notice that the proposition is true for simple k-forms. Moreover, we can extend $*$ to a linear map on $\Omega^k(U)$. Under this definition of $*$, the proposition holds. □

Remark 1. *This gives us a map $* : \Omega^k(U) \to \Omega^{3-k}(U)$ called the* Hodge star. *This satisfies*

$$** = id_{\Omega^k(U)}.$$

5 Vector Calculus

We call functions $f : \mathbb{R}^n \to \mathbb{R}$ *scalar fields* and functions $\mathbf{F} : \mathbb{R}^n \to \mathbb{R}^n$ *vector fields.* In the language of differential forms, scalar fields are 0-forms (i.e. "just functions") and vector fields are 1-forms. We let $n = 3$ for the rest of this section, as $\mathbb{R}^3$ is the domain of vector calculus.

Remark 2. *Let $\mathbf{F} = (F_1, F_2, F_3) : \mathbb{R}^3 \to \mathbb{R}^3$ be a vector field and let $F = F_1 dx + F_2 dy + F_3 dz$. Then*

1. $dF = (\partial_x F_2 - \partial_y F_1)dx \wedge dy + (\partial_x F_3 - \partial_z F_1)dx \wedge dz + (\partial_y F_3 - \partial_z F_2)dy \wedge dz.$
2. $*F = F_1 dy \wedge dz + F_2 dz \wedge dx + F_3 dx \wedge dy$

Definition 6. *For a scalar field f we define the* gradient *∇f as*

$$\nabla f := \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$$

and we note that this is analogous to

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i.$$

Definition 7. *Note that we can write* $\mathbf{F}$ *as* $\mathbf{F} = (F_1, F_2, F_3)$. *We define the* divergence *of* $\mathbf{F}$ *as*

$$Divf := \frac{\partial F_1}{\partial x} + \frac{\partial F_1}{\partial y} + \frac{\partial F_1}{\partial z}.$$

Now define $\omega = F_x dx + F_y dy + F_z dz$. Then

$$Divf = *d * \omega.$$

Definition 8. *We define the* curl *of* F *by*

$$CurlF = \begin{pmatrix} \partial_y F_z - \partial_z F_y \\ \partial_z F_x - \partial_x F_z \\ \partial_x F_y - \partial_y F_x \end{pmatrix}$$

and note that $CurlF = *dF$.

Proposition 5. *(Fundamental identities) For a scalar field* f *and vector field* F *we have*

1. *Curl of gradient is zero:* $Curl(Gradf) = 0$.
2. *Divergence of curl is zero:* $Div(Curl(F)) = 0$.

Proof. First note that

$$\text{Curl}(Grad(F)) = *(d\text{Grad}F) = *(d(dF)) = 0$$

by $d^2 = 0$. Secondly, note that

$$\text{Div}(\text{Curl}(F)) = *(d(*\text{Curl}F)) = *(d(*(*dF))) = *(d^2F) = 0$$

again by $d^2 = 0$. □

Suppose U is a domain that the f and F are defined on. The first identity is equivalent to

$$\Omega^0(U) \xrightarrow{\text{Grad}} \Omega^1(U) \xrightarrow{\text{Curl}} \Omega^1(U)^* \to 0$$

being exact and the second is equivalent to

$$\Omega^1(U) \xrightarrow{\text{Curl}} \Omega^1(U)^* \xrightarrow{\text{Div}} \mathbb{R} \to 0$$

being exact.

Definition 9. *For vector* $v, w \in \mathbb{R}^3$, *define*

$$v \times w = \star(v \wedge w).$$

Remark 3. *Consider* $v \times w$. *Then we get* $\langle v, v \times w\rangle = \langle v, \star(v \wedge w)\rangle = \langle v, v \wedge w\rangle\, Vol$

6 Minkowski Space-Time

We define so-called *Minkowski space-time* as $\mathbb{R}^4$, equipped with the inner product defined by

$$\langle (t, x, y, z), (t', x', y', z')\rangle = -tt' + xx' + yy' + zz'$$

7 Electro-magnetism

8 Next steps

Algebraic Topology

We begin with a brief and informal discussion of homotopy, followed by basic treatment of homology. While algebraic topology is facinating, our main goal is to develop computational tools that will help with geometric calculations. We let X be a "sufficiently nice" path connected space.

9 Basic homotopy

Definition 10. *A* path *in* X *is a continuous function* $\gamma : [0,1] \to X$. *We say two paths* γ_1 *and* γ_2 *are homotopic if there is a function* $F : [0,1] \times [0,1] \to X$ *such that* $F(0,t) = \gamma_1(t)$ *and* $F(1,t) = \gamma_2(t)$ *for all* t. *We call such an* F *a* homotopy. *The set* $\pi_1(X, b)$ *defined to be the set of paths* γ *where* $\gamma(0) = \gamma(1) = b$, *modulo homotopy. We can concatenate two paths through* b *in an obvious way. Indeed, if* γ_1 *and* γ_2 *are paths through* b, *then we can define*

$$(\gamma_1 * \gamma_2)(t) = \begin{cases} \gamma_1(2t) & \textit{if } 0 \le t \le 1/2 \\ \gamma_2(2t) & \textit{if } 1/2 < t \le 1 \end{cases}.$$

Proposition 6. $\pi_1(X, x_0)$ *is a group.*

Proof. Straightforward. □

Proposition 7. *Let* $f : X \to Y$ *be a continuous map, and fix* $x_0 \in X$. *Then there is an induced group homomorphism* $\pi_1(f, x_0) : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$

Proof. Let γ represent a loop in X based at x_0. Then the curve $\eta : [0,1] \to Y$ given by $\eta(t) = f(\gamma(t))$ is a curve in Y based at $f(x_0)$. Moreover, if γ_1 and γ_2 are two curves in X based at x_0 then ... □

Let us compute some examples, where we first give an intutive argument, followed by a rigorous one.

Example 1. *Firstly, consider the unit circle in the complex plane $S^1 \subset \mathbb{C}$, and a basepoint x_0. Basically, loops through x_0 are characterized by their* winding number*. I am using the term a bit loosely here. This characterization is homomorphic. Indeed, if γ_1 and γ_2 have winding numbers r and s, we expect $\gamma_1 * \gamma_2$ to have a winding number of $r + s$. Moreover, we can have negative winding numbers. We are justified by intution to say that $\pi_1(S^1, x_0) \cong \mathbb{Z}$. To prove it, let $\gamma : [0,1] \to S^1$ be a path, and let $\rho : [0,1] \to S^1$ be the function $\rho(t) = e^{2\pi i t}$. Then we can lift γ to a path $\tilde{\gamma} : [0,1] \to \mathbb{R}$, where $\rho(\tilde{\gamma}(t)) = \gamma(t)$ for all t. Then the map $\pi_1(S^1, 1) \to \mathbb{Z}$ given by $\gamma \mapsto \tilde{\gamma}(1)$ is a group isomorphism. So we have $\pi_1(S^1) \cong \mathbb{Z}$.*

Example 2. *As a second example, consider the sphere S^2 and fix a point $p \in S^2$. Then any two loops through p are homotopic to the identity loop $t \mapsto p$. Thus, $\pi_1(S^2) = 0$.*

Lemma 1. *For path connected spaces X and Y and $a \in X, b \in Y$ we have*

$$\pi_1(X \times Y, (a,b)) \cong \pi_1(X, a) \times \pi_1(Y, b)$$

Example 3. *As a final example, let us consider the fundamental group of the torus $\mathbb{T} = S^1 \times S^1$, the donut shape in $\mathbb{R}^3$. Intuitively, we know that there are loops going through the hole, and loops just on the surface - which are indepdendent. Therefore, we are justified in believing that $\pi_1(\mathbb{T})$ has two generators. So we have*

$$\pi_1(\mathbb{T}) = \mathbb{Z} \times \mathbb{Z}$$

as our intuition suggests.

We finish the section with a definition of the homotopy groups $\pi_1, \pi_2, \ldots$
which is consistent with our definition of π_1.

Definition 11. *Let X be a path connected space and let $a \in X$. We define the n-th* homotopy group *$\pi_n(X, a)$ to be the set of continuous functions $S^n \to X$, modulo homotopy, and where the group operation is loop concatentation.*

10 Singular Homology

11 Singular Cohomology

Manifolds

12 Smooth Manifolds

Definition 12. *A n-*dimensional topological manifold *M is a second countable, Hausdorff toplogical space such that for every $x \in M$ there is a neighborhood U of x such that U is homeomorphic to $\mathbb{R}^n$. A k-*smooth atlas *on M is a collection $\{(U_\alpha, \phi_\alpha)\}$ such that (a) $\{U_\alpha\}$ covers M and for each α, β, the function $\varphi_\beta \circ \varphi_\alpha^{-1}$ restricted to $\varphi(U_\alpha \cap U_\beta) \to \mathbb{R}^n$ is C^k. An n-*dimensional C^k manifold *is an n-dimensional topological manifold equipped with a C^k-smooth atlas. We say M is a* ***smooth manfiold*** *if $k = \infty$.*

Proposition 8. *If M is a manifold, then M has a countable atlas.*

Proof. See [Spivak 1]. □

Let M and N be smooth manifolds of dimensions m and n, respectively, and let $f : M \to N$ be a function. Let $x_0 \in M$. Let (U, φ) be a chart containing x_0 and let (V, ψ) be a chart containing $f(x_0)$. Then the function $\psi \circ f \circ \varphi^{-1}$ is a function from $\varphi(U) \subset \mathbb{R}^m$ to $\psi(V) \subset \mathbb{R}^n$, and we say f is *C^k-smooth* at x_0 if the map $\psi \circ f \circ \varphi^{-1}$ is smooth at $\varphi(x_0)$. It is worth noting that this definition is independent of the charts chosen. Indeed, if (U', α) and (V', β) are two other charts of x_0 and $f(x_0)$, respectively, then by the fact that all charts are diffeomorphism, we get that $\beta \circ f \circ \alpha^{-1}$ is also smooth.

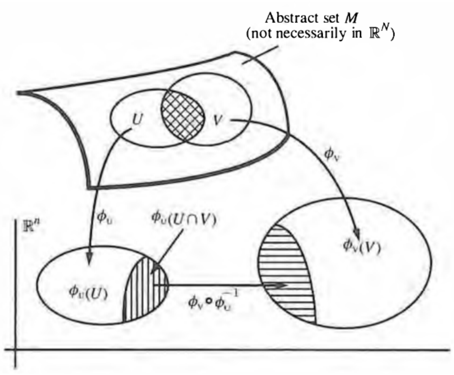

Figure 1: Manifold Charts

If (U, ϕ) is a coordinate chart on M, then there exist smooth functions $\phi_1, \ldots, \phi_n : M \to \mathbb{R}$ such that for every $x \in M$, we have $\phi(x) = (\phi_1(x), \ldots, \phi_n(x))$. We call the ϕ_i's local-coordinates, and denote them by $x^i = \phi_i(x)$.

12.1 Tangent vectors

In this section we introduce the notion of a tangent vector.

Definition 13. *Let $x_0 \in M$. We say two curves $\gamma_1, \gamma_2 : (-1, 1) \to M$ which satisfy $\gamma_1(0) = \gamma_2(0) = x_0$ are equivalent if for every $f \in C^\infty(M)$, we have*

$$\frac{d(f \circ \gamma_1)}{dt}(0) = \frac{d(f \circ \gamma_2)}{dt}(0).$$

At tangent vector *to x_0 is such an equivalence class and we denote all such equivalence classes by T_pM, which we call the* tangent space *at x_0.*

We recall that a *derivation of M at $p \in M$* is an $\mathbb{R}$-linear map $D : C^\infty(M) \to \mathbb{R}$ such that for all $f, g \in C^\infty(M)$ we have $D(fg) = D(f)g(p) + f(p)D(g)$. We denote the vector space of derivations by $\mathrm{Der}_p(C^\infty(M))$.

Proposition 9. *Let $(U, \phi) = (U, x^1, \ldots, x^n)$ be a coordinate chart around p. For $1 \le i \le n$, let $\partial_i = \dfrac{\partial}{\partial x^i}(p)$. Then $\{\partial_1, \ldots, \partial_n\}$ is a basis of $Der_p(C^\infty(M))$.*

Proof. We first show that $\{\partial_1, \ldots, \partial_n\}$ are linearly independent. Indeed, if $a^i\partial_i = 0$, then for $1 \le j \le n$, we can define $f_j : M \to \mathbb{R}$ by $f = x^j$. Then $a^i\partial_i f_j = a^j = 0$. Thus $\partial_1, \ldots, \partial_n$ are linearly indepdendent.

To show that they span $\mathrm{Der}_p(C^\infty(M))$, let $f \in C^\infty(M)$ and consider its Taylor expansion around p,

$$f(x) = f(p) + \sum_{i=1}^{n} \frac{\partial f}{\partial x^i}(x^i - p^i) + R(x)$$

where $R(x)$ consists of higher order terms. Now let $D \in \mathrm{Der}_p(C^\infty(M))$. Then as $D(\text{const}) = 0$ we get

$$D(f) = \sum_{i=1}^{n} a^i \frac{\partial f}{\partial x^i} \qquad a^i = D(x^i).$$

So $\partial_1, \ldots, \partial_n$ span $\mathrm{Der}_p(C^\infty(M))$ as claimed. □

For a tangent vector $[\gamma] \in T_pM$, we define its derivation by

$$D_\gamma(f) = \frac{d(f \circ \gamma)}{dt}(0) \qquad f \in C^\infty(M).$$

Corolary 1. *The map*

$$T_pM \to Der_p(C^\infty(M)) \qquad [\gamma] \mapsto D_\gamma$$

is a vector space isomorphism.

Proof. The map is injective by the definition of the tangent space. The map is surjective by the proof of the above theorem. □

12.2 Bundles

Definition 14. *A* fiber bundle *with fiber F consists of a continuous surjection $\pi : E \to M$ such that for every $x \in M$ there exists an open set U containing x and a homeomorphism $\rho : \pi^{-1}(U) \to U \times F$ such that $\pi = pr_U \circ \rho$.*

Definition 15. *If F is a real finite dimensional vector space, we say π is a* vector bundle.

Definition 16. *A section of a bundle $\pi : E \to M$ is a smooth map $s : M \to E$ such that $\pi \circ s = id_M$. We denote the space of sections of E by $\Gamma(E)$.*

Definition 17. *Given an n-dimensional smooth manifold M, we define its* tangent bundle *TM as*

$$TM = \bigsqcup_{p \in M} T_pM$$

together with a projection map $\pi : TM \to M$ given by $\pi(v_p) = p$. We define a manifold structure on TM as follows. Let (U, φ) be a chart in M. Then for every $p \in U$, we can write $v_p = \sum v^i \dfrac{\partial}{\partial x^i}(p)$. Then we can define a chart $(\pi^{-1}(M), \tilde{\varphi})$ by

$$\tilde{\varphi}(v_p) = (\varphi(p), (v^1, \ldots, v^n)).$$

This makes TM into a $2n$-dimensional manifold.

Definition 18. *We define the cotangent bundle as*

$$T^*M = \bigsqcup_{p \in M} T_pM^*$$

Definition 19. *For $r, s \ge 0$, define the (r, s) tensor bundle as*

$$T^{r,s}M = TM^{r\otimes} \otimes T^*M^{s\otimes}.$$

Definition 20. *A vector field X is a member of $\Gamma(TM)$.*

Definition 21. *A k-*form *is a member of $\Gamma(\Lambda^k T^*M)$. We also denote this by $\Omega^k(M)$.*

Vector Bundle	Section	Section Notation
Tangent Bundle TM	Vector Fields	$\mathfrak{X}(M)$
Alternating Bundle $\Lambda^k TM$	k-forms	$\Omega^k(M)$
Tensor Bundle $T^{r,s}M$	Tensors	$\mathfrak{T}^{r,s}(X)$

12.3 Basics of Vector Fields

We outline the basic theory of vector fields.

Lemma 2. *(Picard-Lindelof) Let $U \subset \mathbb{R}^n$ be open, and let $f : U \to \mathbb{R}$ be a C^1 function. For any $x_0 \in U$, the initial value problem*

$$\frac{dx}{dt} = f(x) \qquad x(0) = x_0$$

has a unique smooth solution $x : (-1, 1) \to U$.

Proposition 10. *Let X be a vector field on M and let $p_0 \in M$. Then there is a unique solution $\Phi_t^X(p_0) : (-1, 1) \to M$ of the differential equation*

$$\frac{d\Phi_t^X(p_0)}{dt} = X(\Phi_t^X(p_0)).$$

Proof. This is immediate from the Picard-Lindelof theorem. ☐

12.4 One-forms

Recall that a k-form is a member of $\Gamma(\Lambda^k T^*M)$. But if V is a vector space, then it is common to identify $\Lambda^1 V$ with its dual V^*. Hence we view one-forms as members of $\Gamma(T^*M) = \Omega^1(M)$.

Definition 22. *Fix $p \in M$ and let $\{\frac{\partial}{\partial x^i}(p) : 1 \le i \le n\}$ be a basis of T_pM. We define its dual basis, i.e. the basis of T_pM^*, as $\{(dx^i)_p : 1 \le i \le n\}$, i.e.*

$$(dx^i)_p \frac{\partial}{\partial x^j}(p) = \delta_{ij}.$$

As we can define $(dx^i)_p$ for each $p \in M$, each dx^i is a legimate one-form, i.e. a section of T^*M.

Definition 23. *Let $f : M \to \mathbb{R}$ be smooth and fix $p \in M$. Then we define the linear map $(df)_p : T_pM \to \mathbb{R}$ by*

$$(df)_p(X_p) = X_p(f).$$

Again, df is a section of T^*M.

Proposition 11.

$$df = \sum_{i=1}^n \frac{\partial f}{\partial x^i} dx^i.$$

Proof. Suppose $X = a^i \partial_i$. Then $(df)(X) = X(f) = a^i \partial_i(f)$. Moreover, $\partial^i(f) dx^i X = a^i \partial_i(f)$ which proves the statement. ☐

12.5 k-forms

Proposition 12. *Every k-form $\omega \in \Omega^k(M)$ has the form*

$$\omega = \sum \omega_{i_1,\dots,i_k} dx^{i_1} \wedge \cdots \wedge dx^{i_k}$$

where $\omega_{i_1,\dots,i_k} \in C^\infty(M)$.

Proof. Recall that members of $\Omega^k(M)$ are functions $\omega : M \to \Lambda^k TM$. So if $p \in M$, then $\omega(p)$ is a multilinear map $(T_pM)^k \to \mathbb{R}$. Now if $\frac{\partial}{\partial x^i}(p)$, $1 \le i \le n$ be a basis of T_pM, and let dx^i, $1 \le i \le n$ be its dual basis. By multilinear algebra, we know $\omega(p)$ has the form

$$\omega(p) = \sum_{1 \le i_1 < \cdots < i_k \le n} \omega_{i_1,\dots,i_k}(p) dx^{i_1} \wedge \cdots dx^{i_k}$$

where the sum ranges over all subsets of $\{1, \dots, n\}$ of cardinality k, and $\omega_{i_1,\dots,i_k} \in C^\infty(M)$ for each $\{i_1, \dots, i_k\}$. ☐

Remark 4. *We can evaluate k-forms as follows. Consider the simple k-form*

$$f dx^{i_1} \wedge \cdots \wedge dx^{i_k} \qquad f \in C^\infty(M).$$

Then if $X_1, \dots, X_k \in \mathfrak{X}(M)$, then

$$(f dx^{i_1} \wedge \cdots \wedge dx^{i_k})(X_1, \dots, X_k) = f \cdot \det \begin{pmatrix} (dx^{i_1})(X_1) & \cdots & (dx^{i_1})(X_k) \\ \vdots & \cdots & \vdots \\ (dx^{i_k})(X_1) & \cdots & (dx^{i_k})(X_k) \end{pmatrix} = f \det(x_{ij}) \qquad X_i = (x_1, \dots, x_n).$$

12.6 The exterior derivative

We are now ready to define the differential of a function, which is the analog of the Jacobian for manifolds.

Definition 24. *Let $f : M \to N$ be a smooth map of manifolds, and fix a $p \in M$. Let v_a be a member of T_pM. Then if $g \in C^\infty(N)$ then we can define the* pushforward *or* differential *$f_{*,p} : T_pM \to T_{f(p)}N$ by*

$$f_{*,p}(X)(g) := X(g \circ f).$$

As we have this for each p, we get a map between tangent bundles

$$f_* : TM \to TN$$

which restricts to $f_{*,p}$ on each T_pM. We also denote f_* by df.

Proposition 13. *f is smooth if and only if f_* is smooth.*

Proof. ... □

Remark 5. *If $f \in C^\infty(M)$ and $\alpha \in \Omega^k(M)$, then we define $f \wedge \alpha = f \cdot \alpha$ (usual multiplication). This makes the below proposition true.*

We now extend d from $C^\infty(M)$ to $\Omega^\bullet(M)$.

Proposition 14. *For each $k \geq 0$, there is a unique derivation $d : \Omega^k(M) \to \Omega^{k+1}(M)$ such that*

1. *d is linear*
2. *$d^2 = d \circ d = 0$*
3. *If $\omega \in \Omega^k(M)$ and $\eta \in \Omega^l(M)$ then $d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^k \omega \wedge d\eta$*

Proof. We already have a map $d : C^\infty(M) \to \Omega^1(M)$. Let us use rule (3) to expand such a d on k-forms. Indeed, for a set $I \subset \{1, \dots, n\}$ of cardinality k and a function $f \in C^\infty(M)$, we can use rules (1), (2) and (3) together to get

$$d(f dx^I) = df \wedge dx^I = \sum_{i \notin I} \frac{\partial f}{\partial x^i} dx^i \wedge dx^I.$$

Which completes the proof. □

Definition 25. *The above definition of d gives us the co-chain complex*

$$\cdots \to \Omega^k(M) \xrightarrow{d_k} \Omega^{k+1}(M) \xrightarrow{d_{k+1}} \cdots$$

from which we obtain the de Rham cohomology groups

$$H^k_{DR}(M) = \ker(d_n)/im(d_{n-1}).$$

Proposition 15. *Let $\psi : M \to N$ be a smooth map of manifolds. Then there is a unique graded algebra homomorphism $\psi^* : \Omega(N) \to \Omega(M)$ such that*

1. *For smooth functions $f : N \to \mathbb{R}$, $\psi^* f = f \circ \psi : M \to \mathbb{R}$.*
2. *For one forms $\alpha : T_pM^* \to \mathbb{R}$, we have $\psi^*\alpha = \alpha \circ d\psi$*
3. *$\psi^*(\alpha \wedge \beta) = \psi^*(\alpha) \wedge \psi^*(\beta)$.*

12.7 The Lie Derivative

Definition 26. *Let M be a smooth manifold of dimension n. Let X be a vector field on M and let Y be a tensor field. We define*

$$L_X Y = \left.\frac{d}{dt}\right|_0 \Phi_t^{X*} Y.$$

Definition 27. *Let $X \in \mathfrak{X}(M)$. We define the map*

$$\iota_X : \Omega^k(M) \to \Omega^{k-1}(M) \qquad \omega \in \Omega^k(M) \quad \iota_X(\omega) = \omega(X, -).$$

Lemma 3. *(Straitening theorem for vector fields) Let $X \in \mathfrak{X}(M)$. Then there is a coordinate chart $(U, x^1, \dots, x^n)$ of M such that $X = \dfrac{\partial}{\partial x^1}$ in local coordinates.*

Remark 6. *We wish to analyze $L_X\omega$ where X is a vector field and ω is a k-form. Let $(U, x^1, \dots, x^n)$ be the local coordinates. Let $\omega = f dx^{i_1} \wedge \cdots \wedge dx^{i_k}$ be a simple k-form. We can re-order $x^2, \dots, x^n$ such that ω necessarily has the form*

$$\omega = f dx^1 \wedge \cdots \wedge dx^k$$

or

$$\omega = f dx^2 \wedge \cdots \wedge dx^{k+1}.$$

Proposition 16. *(Cartan's magic formula) Let X be a vector field and let ω be a k-form. Then*

$$L_X \omega = d\iota_X \omega + \iota_X d\omega.$$

Proof. Choose local coordinates $x^1, \ldots, x^n$ such that $X = \partial/\partial x^1$. There are two cases.

First, assume $\omega = f dx^1 \wedge \cdots \wedge dx^k$. Then

$$\iota_X \omega = f dx^2 \wedge \cdots \wedge dx^k$$

so

$$d\iota_X \omega = \frac{\partial f}{\partial x^1} dx^1 \wedge \cdots \wedge dx^k + \sum_{i=k+1} \frac{\partial f}{\partial x^i} dx^i \wedge dx^2 \cdots \wedge dx^k.$$

Moreover,

$$d\omega = \sum_{i=k+1}^{n} \frac{\partial f}{\partial x^i} dx^i \wedge dx^1 \wedge \cdots \wedge dx^k$$

so

$$\iota_X d\omega = - \sum_{i=k+1}^{n} dx^i \wedge dx^2 \wedge \cdots \wedge dx^k$$

thus

$$d\iota_X \omega + \iota_X d\omega = \frac{\partial f}{\partial x^1} dx^1 \wedge \cdots \wedge dx^k.$$

In the second case, assume $\omega = f dx^2 \wedge \cdots dx^{k+1}$. Then $\iota_X \omega = 0$. Moreover,

$$d\omega = \frac{\partial f}{\partial x^1} dx^1 \wedge dx^2 \wedge \cdots \wedge dx^{k+1} + \sum_{i=k+2}^{n} \frac{\partial f}{\partial x^i} dx^i \wedge dx^2 \wedge \cdots dx^{k+1}.$$

So

$$\iota_X d\omega = \frac{\partial f}{\partial x^1} dx^2 \wedge \cdots \wedge dx^{k+1}.$$

Thus

$$\iota_X d\omega + d\iota_X \omega = \iota_X d\omega.$$

In both cases, the sum $\iota_X d\omega + d\iota_X \omega = L_X \omega$. ☐

Proposition 17. *If X and Y are vector fields, then*

$$L_X Y = XY - YX.$$

Proof. Let $p \in M$ and $f \in C^\infty(M)$. Then

$$L_X Y = \lim_{t \to 0} \frac{\Phi_t^{X*} Y - Y}{t}.$$

Notice that

$$(\Phi_t^{X*} Y)(p)(f) = Y(\Phi_t^X(p))(f \circ \Phi_{-t}^X)$$

and define

$$H(s,t) = Y(\Phi_s^X(p))(f \circ \Phi_t^X).$$

We see that

$$L_X Y = \left. \frac{d}{dt} \right|_0 H(t,t) = \partial_x H(0,0) + \partial_y H(0,0).$$

To compute $\partial_x H(0,0)$, we define $g(q) = Y(f)(q)$ so

$$\partial_x H(0,0) = \left. \frac{d}{dt} \right|_{t=0} g(\Phi_t(p)) = X(g) = X(Y(f))(p).$$

Similarly, we have

$$\partial_y H(0,0) = \frac{d}{dt} Y(p)(f \circ \Phi_{-t}) = Y(p)(-X(f)) = -Y(X(f))(p).$$

So

$$L_X Y = XY - YX.$$

☐

For vector fields X and Y, we use the notation

$$L_X Y = [X, Y].$$

Example 4. *A* Lie group *is a group G such that (a) G is a smooth manifold (b) the multiplcation map $G \times G \to G$ is smooth and (c) the inversion map $G \to G$ is smooth.*

Example 5. *Diff(M) is a Lie group.*

12.8 Integration on Manifolds

Remark 7. *We assume we have a basic theory of integration on $\mathbb{R}^n$. We can use the Riemann integral, as our functions are at least continuous. In other words, if $U \subset \mathbb{R}^n$ is open and $f \in C^k(U)$ then the integral $\int_U f$ is well defined.*

Definition 28. *A collection of functions $\phi_\alpha : M \to [0,1]$ is called a* partition of unity *if for every $x \in M$ there is a open set U containing x such that $(\phi_\alpha)|_W = 0$ for all but finitely many α and $\sum_\alpha \phi_\alpha = 1$. We say it is subordinate to an open cover U_α if support$(\phi_\alpha) := \overline{\{x \in M : \phi_\alpha(x) \neq 0\}} \subset U_\alpha$ for every α.*

Proposition 18. *M has a countable atlas $\{U_1, \ldots\}$ and a partition of unity $\{\phi_1, \ldots\}$ subordinate to the atlas.*

Remark 8. *Let $f : M \to \mathbb{R}^m$ be a smooth map. Let (U, ϕ) be a coordinate chart on M. Then $\phi^* f := f \circ \phi^{-1} : \mathbb{R}^m \to M \to \mathbb{R}^n$ is smooth. We call this the* pullback of f by ϕ.

Definition 29. *Let $\{(U_k, \psi_k : U_k \xrightarrow{\cong} V_k \subset \mathbb{R}^n)\}$ be a countable atlas of M and let $\{\phi_k : M \to [0,1]\}$ be a partition of unity subordinate to the atlas. Let ω be a compactly supported n-form of M. We define the* integral of ω on M *by*

$$\int_M \omega = \sum_{k=1}^{\infty} \int_{V_k} (\psi_k^{-1})^* \omega.$$

Theorem 1. *(Stoke's Theorem) Let M be a real orientable smooth manifold of dimension n with boundary, and let $\omega \in \Omega^{n-1}(M)$ be a differential form with compact support. Then*

$$\int_M d\omega = \int_{\partial M} \omega.$$

Proof. Let $\omega \in \Omega^{n-1}(M)$. Decompose ω as

$$\omega = \sum_{i=1}^{n} (-1)^{i-1} f_i dx^1 \wedge \cdots \widehat{dx^i} \wedge \cdots dx^n.$$

Then

$$d\omega = (\sum_{i=1}^{n} \frac{\partial f_i}{\partial x^i}) dx^1 \wedge \cdots \wedge dx^n.$$

□

12.9 Poincare Lemma

Remark 9. *Let $F : M \to N$ be a smooth map and let $\omega \in \Omega^k(M)$. Then*

$$F^* \omega = \omega \circ dF$$

and moreover

$$(F^* \omega)_x (v_1, \ldots, v_n) = \omega_{F(x)}((dF)_p v_1, \ldots, (dF)_p v_n).$$

Definition 30. *A set $U \subset \mathbb{R}^n$ is said to be* star shaped *if there exists $x_0 \in U$ such that for every $y \in U$, the line segment $L(x_0, y) = \{t x_0 + (1-t) y : 0 \leq t \leq 1\}$ is a subset of U.*

Proposition 19. *Let $U \subset \mathbb{R}^n$ be an open star-shaped set. Then*

$$H^k_{DR}(U) = \{0\}$$

for all $k \geq 1$.

Proof. Let ω be a k-form such that $d\omega = 0$. Then

$$\mathcal{L}_X \omega = \left. \frac{d}{dt} \right|_{t=0} \Phi_t^* \omega$$

and more generally,

$$\frac{d}{dt} \Phi_t^* \omega = \Phi_t^* (\mathcal{L}_X \omega).$$

Therefore,

$$\Phi_1^* \omega - \Phi_0^* \omega = \int_0^1 (\frac{d}{dt} \Phi_t^* \omega) dt = \int_0^1 (\Phi_t^* \mathcal{L}_X \omega) dt.$$

By Cartan's identity, we know

$$\mathcal{L}_X \omega = \iota_X d\omega + d\iota_X \omega$$

so

$$\int_0^1 \Phi_t^* (\iota_X d\omega + d\iota_X \omega) dt = d\left(\int_0^1 (\Phi_t^* \iota_X \omega) dt \right) + \int_0^1 (\Phi_t^* \iota_X d\omega) dt.$$

Now define $K : \Omega^k(U) \to \Omega^{k-1}(U)$ by

$$K\omega = \int_0^1 (\Phi_t^* \iota_X \omega) dt.$$

Then the above equations imply that

$$Kd + dK = \mathrm{id}_{\Omega^k(U)}.$$

Since $d\omega = 0$, we get $dK\omega = \omega$, which proves our claim. □

13 Integration on Chains

14 De Rham's Theorem

15 Riemannian Manifolds

16 Symplectic Manifolds

Advanced Theory

17 Algebraic Varieties

17.1 Commutative Rings and Algebraic Sets

All rings, unless stated otherwise are commutative.

Recall that an *integral domain* or *domain* is a ring where $ab = 0$ implies $a = 0$ or $b = 0$. Moreover, if R is a domain and $a \neq 0$, then if $ab = ac$ then $b = c$, i.e. the *law of cancellation* holds in a domain. This is an equivalent definition. We say that the domain is a *principal ideal domain* or *PID* if every ideal is generated by one element.

As an example $\mathbb{Z}$ is a PID. Indeed, if I is a non-zero ideal, then we can choose elements a and b in I such that $b > a > 1$. Then we can write $b = qa + c$ for some integers q and c satisfying $q \geq 0$ and $0 \leq c < a$. As c is non-negative and a is the smallest positive integer in I, we see that $c = 0$, i.e. $b = qa$. This ring is also a *unique factorization domain*, which is a ring in which each non-zero element decomposes as a unique product of primes.

We call a non-zero element a of R a *unit* if there exists some b such that $ab = 1$. Ideals generated by units are the entire ring. A non-unit which is not the product of any two non-units is called an *irreducible element.* A *unique factorization domain* or *UFD* is a ring in which every non-zero, non-unit element is a unique product of irreducible elements. Hence, $\mathbb{Z}$ is a UFD. We also have the following well-known fact, which is easy to prove.

Theorem 2. *If R is a UFD then so is $R[X]$ and consequently so is $R[X_1, \ldots, X_n]$.*

We say R is *Noetherian* if every ideal is finitely generated. We have the following important theorem.

Theorem 3. *If R is Neotherian, then so is $R[X]$ and consequently so is $R[X_1, \ldots, X_n]$.*

We use the notation $R[\mathbf{X}]$ instead of $R[X_1, \ldots, X_n]$ when it is convenient and it does not cause confusion.

There are two proofs. We give a sketch of both. The first way is to take an ideal I in $R[X]$ and take the set of leading coefficients in I which is a finitely generated ideal in R. We call this ideal J. The idea of the proof is to decompose J into a sum which has a nice pre-image in $R[x]$.

The better way of approaching this is to use the concept of a *Grobner basis.* Unfortunately, I could not find a precise definition, but a Grobner basis is basically a finite generating set of I with "nice algorithmic properties". Thankfully, the exact definition does not matter for us. Basically, if $F = \sum a_i X_1^{i_i} \cdots X_k^{i_k}$, and $G_1, \ldots, G_m$ are a set of polynomials in $R[X]$, then we can write $F = \sum A_i G_i + C$, where $\deg C < \deg F$. This can be done by a sort of Euclidean algorithm using lexicographic ordering, called *Bucherberg's Algorithm.*

Given a subset X of R^n, we define its ideal $I(X)$ to be the set of polynomials $f \in R[\mathbf{X}]$ are zero on X. Similarly, for an ideal I of R, we define its *locus* to be the set of points $p \in R^n$ such that $fp = 0$ for all $f \in I$. In general, these maps are not inverses of each other, but we will get to their relationship quite soon. However, we can see some interesting properties. First, we say an algebraic set is *irreducible* if it is not a union of two non-empty algebraic sets, where the two sets in the union are unequal. It is easy to show that V is irreducible if and only if $I(V)$ is prime. Moreover, by Hilbert's basis theorem, we have

Proposition 20. *Every algebraic set is a finite intersection of irreducible algebraic sets.*

Proof. If I is minimally generated by irreducible polynomials $F_1, \ldots, F_k$, then $X = Z(F_1) \cap \cdots \cap Z(F_k)$. □

Example 6. $\mathbb{A}^1$ *is irreducible.*

Example 7. *Let $f \in k[x_1, \ldots, x_n]$. Then f is irreducible if and only if the ideal (f) is prime, if and only if $Z(f)$ is irreducible.*

Note that if f is reducible if and only if f has a non-trivial factorization $f = gh$. In this case, $Z(g)$ and $Z(h)$ would be proper subsets of $Z(f)$, and we would also have $Z(f) = Z(g) \cup Z(h)$. So we have shown that if f is reducible, then $Z(f)$ can be decomposed as two non-empty algebraic sets.

On the other hand, suppose $Z(f) = X \cup Y$ for non-empty algebraic sets X and Y.

Observation 1. *X is irreducible if and only if $I(X)$ is prime.*

We continue to get more interesting properties of algebraic sets. Suppose $I \subset J$ are ideals. Then $V(J) \subset V(I)$. The situation is simple when k is algebraically closed. If I is a proper ideal of $k[\mathbf{X}]$, then there is some maximal ideal J containing I. But we know J is generated by $\langle x_1 - a_1, \dots, x_n - a_n \rangle$. So $a = (a_1, \dots, a_n) \in V(I)$. Hence

Proposition 21. *If k is algebraically closed and I is a proper ideal of $k[\mathbf{X}]$ then $V(I) \neq 0$.*

Next we define the radical of an ideal I as

$$\mathrm{rad}(I) = \{f \in R[\mathbf{X}] : f^r \in I \text{ for some } r > 0\}.$$

When k is an algebraically closed field, I claim that $I(V(I)) = \mathrm{rad}(I)$. If we had this, then this would allow us to reduce a generating ideal of an algebraic set to a minimal one. Showing that $\mathrm{rad}(I) \subset I(V(I))$ is straightforward. For the other direction, suppose $f \in I(V(I))$. Let $f_1, \dots, f_k$ be the generators of I. We wish to find some r such that $f^r = \sum a_i f_i$ where a_i are members of $k[X_1, \dots, X_n]$. Then the ideal J generated by $f_1, \dots, f_k$ and $1 - x_{n+1} f$ is the entire ring, since if it were not, then $V(J)$ would have at least one element, which it does not, because f vanishes whenever each f_i does. Thus one can write

$$1 = g_1 f_1 + \cdots + g_k f_k + h(1 - x_{n+1} f)$$

for some g_i and h in $k[x_1, \dots, x_{n+1}]$. Substituting $1/f(x_1, \dots, x_n)$ for x_{n+1} results in the equation

$$1 = \sum g_i(x_1, \dots, x_n, \frac{1}{f(x_1, \dots, x_n)}) f_i(x_1, \dots, x_n).$$

We can multiply both sides by a high enough power of f to remove f from the denominator of the rational expression on the right hand side, yielding

$$f^r = \sum c_i f_i$$

for some c_i in I. Thus we have the following theorem

Theorem 4. *(Hilbert's Nullstellensats) $I(V(I)) = rad(I)$.*

This important theorem characterizes gives us a one-to-one correspondence between the radical ideals of $\mathfrak{r}$ and the algebraic sets in $\mathbb{A}^n$. We can go even further. We say $\mathfrak{r}$ is *reduced* if it contains no non-zero nilpotent elements.

Proposition 22. *$\mathfrak{r}/\mathfrak{i}$ is reduced if and only if $\mathfrak{i}$ is a radical ideal.*

Proof. Suppose $\mathfrak{r}/\mathfrak{i}$ is reduced. To show that $\mathfrak{i}$ is radical, suppose $x \in \mathfrak{r}$ is such that $x^n \in \mathfrak{i}$ for some $n \geq 1$. We aim to show that $x \in \mathfrak{i}$. We can safely assume $n > 1$. Then let m be the smallest positive integer such that $2m > n$. Then $y = x^m$ is a nonzero nilpotent element in $\mathfrak{r}$, which is impossible.

Secondly, if $\mathfrak{i}$ is radical, then if $x \in \mathfrak{r}/\mathfrak{i}$ is nilpotent, then $x^2 \in \mathfrak{i}$, so $x \in \mathfrak{i}$. ☐

Thus we have a one-to-one correspondence between the following sets

$$\text{radical ideals of } \mathfrak{r} \longleftrightarrow \text{ reduced quotients of } \mathfrak{r} \longleftrightarrow \text{ alg. sets}$$

The reader may have noticed, or already knows that the "reduced quotient" is just a ring $k[x_1, \dots, x_n]/I(X)$ where X is an algebraic set. We call this the coordinate ring, and delay a full study to the next section.

Let us sketch out some more properties of algebraic sets before we finish this section. We will finish with giving $\mathbb{A}^n$ a topology, and we will introduce the notion of dimension of algebraic sets.

Proposition 23.
1. *The union of two algebraic sets is algebraic.*
2. *The intersection of two algebraic sets is algebraic.*
3. *$\emptyset$ and $\mathbb{A}^n$ are algebraic.*

Proof. For 1. and 2., we let X and Y be algebraic sets generated by $I_X = (F_1, \dots, F_n)$ and $I_Y = (G_1, \dots, G_m)$. Then note that $X \cup Y$ is generated by the product of the ideals $I_X \cdot I_Y$ and $X \cap Y$ is generated by the sum $I_X + I_Y$. Finally, the ideal of $\emptyset$ is the ring $\mathfrak{r}$ and the ideal of $\mathbb{A}^n$ is $\{0\}$. ☐

Definition 31. *We give $\mathbb{A}^n$ the* Zariski topology*, which defines the open sets as complements of algebraic sets in $\mathbb{A}^n$. We call open sets* quasi-algebraic sets.

As an example, note that the Zariski-closed sets of $\mathbb{A}^1$ are the finite subsets of $\mathbb{A}^1$. The closed sets in $\mathbb{A}^2$ consist of finite sets, lines and curves.

Definition 32. *An* affine variety *is an irreducible affine algebraic set. A* quasi-affine variety *is a an open subset of an affine variety.*

Remark 1. *The Zariski topology is Hausdorff if and only if k is a finite field...*

We define the *Krull dimension* of a ring to be the supremum of lengths of proper chains of prime ideals, and we denote this by $\dim \mathfrak{r}$. For a prime ideal $\mathfrak{p} \subset \mathfrak{r}$, we define its height to be the supremum of its proper chains of prime ideals.

Proposition 24. *If $\mathfrak{r}$ is Noetherian and $\mathfrak{p}$ is a prime ideal of $\mathfrak{r}$, then*

$$\dim \mathfrak{r} = \dim \mathfrak{r}/\mathfrak{p} + \operatorname{ht}(\mathfrak{p}).$$

Proof. Left to the reader. ☐

Definition 33. *If X is a topological space, we define its* dimension *to be the supremum of lengths of strictly descending chains of closed sets in X.*

Proposition 25. *Let X be an algebraic set in $\mathbb{A}^n(k)$. Then the topological dimension of X is equal to the Krull dimension of $A(X)$.*

Proof. Let $X \supset Y_0 \supset Y_1 \supset \cdots$ be a descending chain of closed sets in X, where $Y_i \neq Y_{i+1}$. As each Y_i is closed in X, we have $Y_i = Z(\tilde{I}_i)$ where $\tilde{I}_i = I_i/I(X)$ for some prime ideal I_i in $k[x_1, \ldots, x_n]$. We thus get an ascending chain of ideals $I_i \subset \cdots$ which terminates at the Krull dimension of $A(X)$ ☐

For example, the dimension of $\mathbb{A}^n$ is n.

Proposition 26. *If X is a quasi-affine variety, then* $\dim X = \dim \overline{X}$.

We leave the proof as an exercise to the reader.

Proposition 27. *A variety X in $\mathbb{A}^n$ has dimension $n-1$ if and only if X is generated by a single irreducible polynomial.*

Proof. Suppose X has dimension 1, and let $\mathfrak{p}$ denote its prime ideal. Then by a proposition above, $\mathfrak{p}$ has height 1. By another theorem, this is principle.

Conversely, if X is generated by a principal prime ideal $\mathfrak{p}$, then this ideal has height 1 and proves the theorem. ☐

17.2 Projective algebraic sets

17.3 Local Behaviour

17.4 Varieties

18 Homological Algebra

Let $\mathcal{A}$ be an abelian category. Let **Ab** be the category of abelian groups.

Lets begin by discussing the notion of Hom. For objects A and B in $\mathcal{A}$, we have an abelian group $\operatorname{Hom}(A, B)$, which is the set of morphism from A to B. This is part of the definition of an abelian category (abelian categories are *pre-additive*). Let us functorialize this in the most obvious way.

Let $f : A \to B$ be a morphism, and let D be an object in $\mathcal{A}$. We transform this into a morphism $\operatorname{Hom}(D, A) \to \operatorname{Hom}(D, B)$ as follows. Let $g : D \to A$ be a morphism. Then $fg : D \to B$ is also a morphism. So (excluding technical details) we get a covariant functor $\operatorname{Hom}(D, -)$ from $\mathcal{A}$ to **Ab**.

When D is in the other slot of Hom, we get a different relationship. Let $g : B \to D$ be a morphism. Then $gf : A \to B \to D$ is also a morphism. This gives us a contravariant functor from $\mathcal{A}$ to **Ab**.

Proposition 28. *For any object D in $\mathcal{A}$, Hom$(D, -)$ is left-exact and co-variant, and Hom$(-, D)$ is left-exact and contravariant.*

We say D is *projective* if for every two objects M and N and exact sequence $M \xrightarrow{\varphi} N \to 0$, for every morphism $f : D \to N$, there is a morphism $F : D \to M$ such that the following diagram commutes:

$$\begin{tikzcd}
& D \arrow[d, "f"] \arrow[dl, dashed, "F"'] & \\
M \arrow[r, "\varphi"] & N \arrow[r] & 0
\end{tikzcd}$$

We say D is injective when there is an exact sequence $0 \to M \xrightarrow{\varphi} N$, and for every morphism $f : N \to D$ there is a morphism $g : N \to D$ such that the following diagram commutes:

$$\begin{tikzcd}
& D & \\
0 \arrow[r] & M \arrow[u, "f"] \arrow[r, "\varphi"] & N \arrow[ul, dashed, "g"']
\end{tikzcd}$$

Proposition 29. *D is projective if and only if Hom$(D, -)$ is exact and injective if and only if Hom$(-, D)$ is exact.*

For an object D, we say D is *projective* if $\mathrm{Hom}(D, -)$ is exact and *injective* if $\mathrm{Hom}(-, D)$ is exact. An equivalent characterization of a projective object is as follows. If M and N are objects and $M \xrightarrow{f} N \to 0$ is exact then there is a lift $F : P \to M$ such that the following diagram commutes

Definition 34. *We say a sequence of morphisms* $C_\bullet$

$$0 \to A_0 \xrightarrow{d_1} A_1 \xrightarrow{d_2} \cdots$$

is a cochain complex *if* $d_{n+1}d_n = 0$ *for all* $n \geq 1$. *We define the* n-th cohomology group *to be the quotient* $H^n(C_\bullet) := \ker d_{n+1}/imd_n$.

Note that we can take morphisms between cochain complexes (resp, short exact sequences) in an obvious way so we can form $\mathrm{Ch}(\mathcal{A})$, the category of co-chain complexes. This is also an abelian category.

Proposition 30. *If* $0 \to A \to B \to C \to 0$ *is a short-exact sequence in* $\mathcal{A}$, *then we have a long-exact sequence*

$$\cdots \to H^n(A) \to H^n(B) \to H^n(C) \xrightarrow{\delta_n} H^{n+1}(A) \to \cdots$$

Proof. See [Hatcher] or [Dummit&Foote]. This is an application of simultaneous resolution (aka the horseshoe lemma) and the snake-lemma. □

For an object A, a projective resolution is an exact sequencce

$$\cdots P_n \to \cdots \to P_0 \to A \to 0$$

where each P_i is projective. We say $\mathcal{A}$ has *enough projectives* if every object has a projective resolution.

As an example, note that the category of R-modules has enough projectives. Indeed, if M is an R module generated by a set S. Let F_0 be the free module generated by S and let $\epsilon : F_0 \to M$ be the morphism which restricts to the identity on S. Let K_0 be the kernel of ε. This yields a short exact sequence

$$0 \to K_0 \to F_0 \to M \to 0.$$

Then repeat the same process with K_0. Indeed, let T be a generating set of K_0, let F_1 be the free-module generated by T, and let $\varepsilon' : F_1 \to K_0$ be the map which restricts to the identity on T. Then we get an exact sequence

$$0 \to F_1 \to F_0 \to M \to 0$$

which we can continue inductively.

Let us dig deeper into projective resolutions. Let A and D be objects and let $P_\bullet \to A \to 0$ be a projective resolution of A. We can apply $\mathrm{Hom}(-, D)$, which we know is contravariant to get a co-chain complex $0 \to \mathrm{Hom}(A, D) \to \mathrm{Hom}(P_0, D) \to \cdots$. This complex is not necessarily exact, but it is indeed a complex. Hence. we obtain cohomology groups, but we need to prove that these groups are indepdendent of the choice of resolution. We will finish our first discussion of homological algebra with a proof of this fact. Admittedly this will be pretty sketchy.

Lemma 4. *Let* $P_\bullet \to A$ *be a projective resolution of* A *and let* $Q_\bullet \to A'$ *be an exact co-chain complex. Let* $f : A \to A'$ *be a morphism. Then for each* $n \geq 0$ *there is a morphism* $f_n : P_n \to Q_n$ *such that the following diagram commutes.*

$$\begin{tikzcd}
\cdots \arrow[r] & P_1 \arrow[r] \arrow[d, dashed, "f_1"'] & P_0 \arrow[r] \arrow[d, dashed, "f_0"'] & A \arrow[r] \arrow[d, "f"] & 0 \\
\cdots \arrow[r] & Q_1 \arrow[r] & Q_0 \arrow[r] & A' \arrow[r] & 0
\end{tikzcd}$$

and we say $f_\bullet$ *is a* lift *of* f. *Moreover, any two lifts of* f *are chain-homotopic, i.e. for any other another lift* $g_\bullet : P_\bullet \to Q_\bullet$, *there exists morphisms* $s_n : P_n \to Q_{n+1}$ *such that*

$$d^D_{n+1}s_n + s_{n-1}d^C_n = g_n - f_n$$

and we denote this by $f_\bullet \simeq g_\bullet$.

Proof. As P_0 is projective, and there is a morphism $P_0 \to A \to A'$ there is also morphism from P_0 to Q_0. Call this f_0. Similarly, the map $P_1 \to P_0 \to Q_0$ induces a map $P_1 \to Q_1$ by projectivity. The existence of the other f_n's exist by induction. Finally, we can look at the chain map $f_\bullet - g_\bullet$. We define $s_{-1} = 0$ and obtain a diagonal map $s_0 : P_0 \to Q_1$ by projectivity. The rest can exist by induction. □