

0.1 Homological Algebra

Let \mathcal{A} be an abelian category. Let \mathbf{Ab} be the category of abelian groups.

Lets begin by discussing the notion of Hom. For objects A and B in \mathcal{A} , we have an abelian group $\text{Hom}(A, B)$, which is the set of morphism from A to B . This is part of the definition of an abelian category (abelian categories are *pre-additive*). Let us functorialize this in the most obvious way.

Let $f : A \rightarrow B$ be a morphism, and let D be an object in \mathcal{A} . We transform this into a morphism $\text{Hom}(D, A) \rightarrow \text{Hom}(D, B)$ as follows. Let $g : D \rightarrow A$ be a morphism. Then $fg : D \rightarrow B$ is also a morphism. So (excluding technical details) we get a covariant functor $\text{Hom}(D, -)$ from \mathcal{A} to \mathbf{Ab} .

When D is in the other slot of Hom, we get a different relationship. Let $g : B \rightarrow D$ be a morphism. Then $gf : A \rightarrow B \rightarrow D$ is also a morphism. This gives us a contravariant functor from \mathcal{A} to \mathbf{Ab} .

Proposition 1. *For any object D in \mathcal{A} , $\text{Hom}(D, -)$ is left-exact and co-variant, and $\text{Hom}(-, D)$ is left-exact and contravariant.*

We say D is *projective* if for every two objects M and N and exact sequence $M \xrightarrow{\varphi} N \rightarrow 0$, for every morphism $f : D \rightarrow N$, there is a morphism $F : D \rightarrow M$ such that the following diagram commutes:

$$\begin{array}{ccccc} & & D & & \\ & \swarrow F & \downarrow f & & \\ M & \xrightarrow{\varphi} & N & \longrightarrow & 0 \end{array}$$

We say D is *injective* when there is an exact sequence $0 \rightarrow M \xrightarrow{\varphi} N$, and for every morphism $f : N \rightarrow D$ there is a morphism $g : M \rightarrow D$ such that the following diagram commutes:

$$\begin{array}{ccccc} & & D & & \\ & & \uparrow f & \swarrow g & \\ 0 & \longrightarrow & M & \xrightarrow{\varphi} & N \end{array}$$

Proposition 2. *D is projective if and only if $\text{Hom}(D, -)$ is exact and injective if and only if $\text{Hom}(-, D)$ is exact.*

Definition 1. *We say a sequence of morphisms C_\bullet .*

$$0 \rightarrow A_0 \xrightarrow{d_1} A_1 \xrightarrow{d_2} \dots$$

is a cochain complex if $d_{n+1}d_n = 0$ for all $n \geq 1$. We define the n -th cohomology group to be the quotient $H^n(C_\bullet) := \ker d_{n+1} / \text{im} d_n$.

Note that we can take morphisms between cochain complexes (resp, short exact sequences) in an obvious way so we can form $\text{Ch}(\mathcal{A})$, the category of co-chain complexes. This is also an abelian category.

Proposition 3. *If $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ is a short-exact sequence in \mathcal{A} , then we have a long-exact sequence*

$$\dots \rightarrow H^n(A) \rightarrow H^n(B) \rightarrow H^n(C) \xrightarrow{\delta_n} H^{n+1}(A) \rightarrow \dots$$

Proof. See [Hatcher] or [Dummit&Foote]. This is an application of simultaneous resolution (aka the horseshoe lemma) and the snake-lemma. \square

For an object A , a projective resolution is an exact sequence

$$\dots P_n \rightarrow \dots \rightarrow P_0 \rightarrow A \rightarrow 0$$

where each P_i is projective. We say \mathcal{A} has *enough projectives* if every object has a projective resolution.

As an example, note that the category of R -modules has enough projectives. Indeed, if M is an R module generated by a set S . Let F_0 be the free module generated by S and let $\epsilon : F_0 \rightarrow M$ be the morphism which restricts to the identity on S . Let K_0 be the kernel of ϵ . This yields a short exact sequence

$$0 \rightarrow K_0 \rightarrow F_0 \rightarrow M \rightarrow 0.$$

Then repeat the same process with K_0 . Indeed, let T be a generating set of K_0 , let F_1 be the free-module generated by T , and let $\epsilon' : F_1 \rightarrow K_0$ be the map which restricts to the identity on T . Then we get an exact sequence

$$0 \rightarrow F_1 \rightarrow F_0 \rightarrow M \rightarrow 0$$

which we can continue inductively.

Let us dig deeper into projective resolutions. Let A and D be objects and let $P_\bullet \rightarrow A \rightarrow 0$ be a projective resolution of A . We can apply $\text{Hom}(-, D)$, which we know is contravariant to get a co-chain complex $0 \rightarrow \text{Hom}(A, D) \rightarrow \text{Hom}(P_0, D) \rightarrow \dots$. This complex is not necessarily exact, but it is indeed a complex. Hence, we obtain cohomology groups, but we need to prove that these groups are independent of the choice of resolution. We will finish our first discussion of homological algebra with a proof of this fact. Admittedly this will be pretty sketchy.

Lemma 1. *Let $P_\bullet \rightarrow A$ be a projective resolution of A and let $Q_\bullet \rightarrow A'$ be an exact co-chain complex. Let $f : A \rightarrow A'$ be a morphism. Then for each $n \geq 0$ there is a morphism $f_n : P_n \rightarrow Q_n$ such that the following diagram commutes.*

$$\begin{array}{ccccccc} \cdots & \longrightarrow & P_1 & \longrightarrow & P_0 & \longrightarrow & A \longrightarrow 0 \\ & & \downarrow f_1 & & \downarrow f_0 & & \downarrow f \\ \cdots & \longrightarrow & Q_1 & \longrightarrow & Q_0 & \longrightarrow & A' \longrightarrow 0 \end{array}$$

and we say f_\bullet is a lift of f . Moreover, any two lifts of f are chain-homotopic, i.e. for any other another lift $g_\bullet : P_\bullet \rightarrow Q_\bullet$, there exists morphisms $s_n : P_n \rightarrow Q_{n+1}$ such that

$$d_{n+1}^Q s_n + s_{n-1} d_n^P = g_n - f_n$$

and we denote this by $f_\bullet \simeq g_\bullet$.

Proof. As P_0 is projective, and there is a morphism $P_0 \rightarrow A \rightarrow A'$ there is also morphism from P_0 to Q_0 . Call this f_0 . Similarly, the map $P_1 \rightarrow P_0 \rightarrow Q_0$ induces a map $P_1 \rightarrow Q_1$ by projectivity. The existence of the other f_n 's exist by induction. Finally, we can look at the chain map $f_\bullet - g_\bullet$. We define $s_{-1} = 0$ and obtain a diagonal map $s_0 : P_0 \rightarrow Q_1$ by projectivity. The rest can exist by induction. \square